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Noncommutativity and reparametrization symmetry

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Abstract
We discuss a general method of revealing both space–space and space–time noncommuting structures in various models in particle mechanics exhibiting reparametrization symmetry. Starting from the commuting algebra in the conventional gauge, it is possible to obtain a noncommuting algebra in a nonstandard gauge. The change of variables relating the algebra in the two gauges is systematically derived using gauge/reparametrization transformations.

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1. Introduction

Issues related to noncommutative space–time in field theories [1] have led to deep conceptual and technical problems prompting corresponding studies in quantum mechanics. In this context, an important role is played by redefinitions or change of variables which provide a map among the commutative and noncommutative structures [2–5]. However, there does not seem to be a precise underlying principle on which such maps are based. One of the motives of this paper is to provide a systematic formulation of such maps. In the models discussed here, these maps are essentially gauge/reparametrization transformations.

A general feature indicated by this analysis is the possibility of noncommuting space–space (or space–time) coordinates for models in particle mechanics with reparametrization symmetry. The point to note is that even if the model does not have this symmetry naturally, it can always be introduced by hand as, for instance, in the non-relativistic (NR) free particle. We shall discuss this example in detail and reveal the various noncommuting structures. As other examples, we consider the free relativistic particle as well as its interaction with a background electromagnetic field.

We exploit the reparametrization invariance to find a nonstandard gauge in which the space–time and/or space–space coordinates become noncommuting. In contrast to recent approaches [4], we provide a definite method of finding this gauge. We also show that the variable redefinition relating the nonstandard and standard gauges is a gauge transformation.

In section 2, we discuss how any particle model can be rewritten in a time reparametrization invariant form. This is used to show the occurrence of noncommuting structures in the usual non-relativistic free particle model. The free relativistic particle is analysed in section 3. Here we also analyse the structure of the angular momentum operator in some detail. A gauge-independent expression is obtained, which therefore does not require any central extension in the nonstandard gauge. The interaction of the free relativistic particle with an external electromagnetic field is considered in section 4. Finally, we conclude in section 5.

There are two appendices. In appendix A, we establish the connection between Dirac brackets in the axial and radiation gauges using suitable gauge transformations. In appendix B, we show, in the symplectic formalism, the connection between integral curves and the equations of motion in the time reparametrized version. This also shows how constraints come into the picture in the time-reparametrized formulation.

2. Particle models

Consider the action for a point particle in classical mechanics

$$S[x(t)] = \int_{t_1}^{t_2} \mathrm{d}t\, L\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right). \tag{1}$$

The above form of the action can be rewritten in a time-reparametrized invariant form by elevating the status of time t to that of an additional variable, along with x, in the configuration space as

$$S[x(\tau), t(\tau)] = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, \dot{t} L\left(x, \frac{\dot{x}}{\dot{t}}\right) = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, L_\tau(x, \dot{x}, t, \dot{t}) \tag{2}$$

where

$$L_\tau(x, \dot{x}, t, \dot{t}) = \dot{t} L\left(x, \frac{\dot{x}}{\dot{t}}\right), \qquad \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}\tau} \quad \dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau} \tag{3}$$

and τ is the new evolution parameter that can be taken to be an arbitrary monotonically increasing function of time t. Now the canonical momenta corresponding to the coordinates t and x are given by

$$\begin{aligned} p_t &= \frac{\partial L_\tau}{\partial \dot{t}} = L\left(x, \frac{\dot{x}}{\dot{t}}\right) + \dot{t}\frac{\partial L\left(x, \frac{\dot{x}}{\dot{t}}\right)}{\partial \dot{t}} \\ &= L\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right) - \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial L(x, \mathrm{d}x/\mathrm{d}t)}{\partial(\mathrm{d}x/\mathrm{d}t)} = -H \end{aligned} \tag{4}$$

$$p_x = \frac{\partial L_\tau}{\partial \dot{x}}. \tag{5}$$

As happens for a time-reparametrized theory, the canonical Hamiltonian (using (4) and (5)) vanishes:

$$H_\tau = p_t\dot{t} + p_x\dot{x} - L_\tau = \dot{t}(H + p_t) = 0. \tag{6}$$

As a particular case of (1), we start from the action of a free NR particle in one dimension

$$S = \int \mathrm{d}t \frac{1}{2}m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2. \tag{7}$$

The above form of the action can be rewritten in a time-reparametrized invariant form as in (2),

$$S = \int \mathrm{d}\tau\, L_\tau(x, \dot{x}, t, \dot{t}) \tag{8}$$

where

$$L_\tau(x, \dot{x}, t, \dot{t}) = \frac{m}{2}\frac{\dot{x}^2}{\dot{t}}, \qquad \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}\tau} \quad \dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau} \tag{9}$$

and τ is the new evolution parameter that can be taken to be an arbitrary monotonically increasing function of time t. Now the canonical momenta corresponding to the coordinates t and x are given by

$$p_t = \frac{\partial L_\tau}{\partial \dot{t}} = -\frac{m\dot{x}^2}{2\dot{t}^2} \tag{10}$$

$$p_x = \frac{\partial L_\tau}{\partial \dot{x}} = \frac{m\dot{x}}{\dot{t}} \tag{11}$$

which satisfy the standard canonical Poisson bracket (PB) relations

$$\{x, x\} = \{p_x, p_x\} = \{t, t\} = \{p_t, p_t\} = 0 \qquad \{x, p_x\} = \{t, p_t\} = 1. \tag{12}$$

As happens for a time-reparametrized theory, the canonical Hamiltonian (using (10) and (11)) vanishes:

$$H_\tau = p_t\dot{t} + p_x\dot{x} - L_\tau = 0. \tag{13}$$

Also, the primary constraint in the theory, obtained from (10) and (11), is given by

$$\phi_1 = p_x^2 + 2mp_t \approx 0 \tag{14}$$

where ≈ 0 implies equality in the weak sense [11, 12]. Clearly the space–time coordinate $x^\mu(\tau)$ ($\mu = 0, 1$; $x^0 = t, x^1 = x$), transforms as a scalar under reparametrization:

$$\tau \to \tau' = \tau'(\tau) \qquad x^\mu(\tau) \to x'^\mu(\tau') = x^\mu(\tau). \tag{15}$$

Consequently under an infinitesimal reparametrization transformation ($\tau' = \tau - \epsilon$), the infinitesimal change in the space–time coordinate is given by

$$\delta x^\mu(\tau) = x'^\mu(\tau) - x^\mu(\tau) = \epsilon\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}. \tag{16}$$

The generator of this reparametrization transformation is obtained by first writing the variation in the Lagrangian L_τ (9) under the transformation (16) as a total derivative,

$$\delta L_\tau = \frac{\mathrm{d}B}{\mathrm{d}\tau}, \qquad B = \frac{m\epsilon}{2}\frac{\dot{x}^2}{\dot{t}} \tag{17}$$

Now the generator G is obtained from the usual Noether's prescription as

$$G = p_t\delta t + p_x\delta x - B = \frac{\epsilon\dot{t}}{2m}\phi_1. \tag{18}$$

It is easy to see that this generator reproduces the appropriate transformation (16)

$$\delta x^\mu(\tau) = \{x^\mu, G\} = \epsilon\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{19}$$

which is in agreement with Dirac's treatment [11, 12][1]. Note that x^μ are not gauge invariant variables in this case. This example shows that reparametrization symmetry can be identified with gauge symmetry.

[1] In this treatment, the generator is a linear combination of the first class constraints. Since we have only one first class constraint ϕ_1, the gauge generator is proportional to ϕ_1.

Let us now fix the gauge symmetry by imposing a gauge condition. The standard choice is to identify the time coordinate t with the parameter τ,

$$\phi_2 = t - \tau \approx 0. \tag{20}$$

The constraints (14) and (20) form a second class set with

$$\phi_{ab} = \{\phi_a, \phi_b\} = -2m\epsilon_{ab} \qquad (a, b = 1, 2) \tag{21}$$

where ϵ_{ab} is an anti-symmetric tensor with $\epsilon_{12} = 1$.

The next step is to compute the Dirac brackets (DB) defined as

$$\{A, B\}_{\rm DB} = \{A, B\} - \{A, \phi_a\}(\phi^{-1})_{ab}\{\phi_b, B\} \tag{22}$$

where A, B are any pair of phase-space variables and $(\phi^{-1})_{ab} = (2m)^{-1}\epsilon_{ab}$ is the inverse of ϕ_{ab}. It then follows:

$$\{x, x\}_{\rm DB} = \{p_x, p_x\}_{\rm DB} = 0 \qquad \{x, p_x\}_{\rm DB} = 1. \tag{23}$$

This reproduces the expected canonical bracket structure in the usual $2 - d$ reduced phase-space comprising variables x and p_x only. The DB imply a strong imposition of the second class constraints (ϕ_a). Consistent with this, $\{t, x\}_{\rm DB} = 0$ showing that there is no space–time noncommutativity if a gauge-fixing condition like (20) is chosen. A natural question that arises is whether space–time (or space–space) noncommutativity can be obtained by imposing a suitable variant of the gauge fixing condition (20). Before answering this question, we emphasize that the DB between various gauges should be related by suitable gauge transformations[2]. This idea will be useful.

In the present case, to get hold of a set of variables x', t' satisfying a noncommutative algebra,

$$\{t', x'\}_{\rm DB} = \theta \tag{24}$$

with θ being constant, the same procedure as done (in appendix A) for a free Maxwell theory is adopted. The transformations (16) are written in terms of phase-space variables, after strongly implementing the constraint (20). Then, in component notation,

$$t' = t + \epsilon \tag{25}$$

$$x' = x + \epsilon\frac{{\rm d}x}{{\rm d}\tau} = x + \epsilon\frac{p_x}{m}. \tag{26}$$

Substituting back in the LHS of (24) and using the Dirac algebra (23) for the unprimed variables, fixes ϵ as

$$\epsilon = -\theta p_x. \tag{27}$$

This shows that the desired gauge fixing condition is

$$t' + \theta p_x - \tau \approx 0. \tag{28}$$

Now one can just drop the prime to rewrite (28) as

$$t + \theta p_x - \tau \approx 0. \tag{29}$$

Expectedly, a direct calculation of the Dirac bracket in this gauge immediately reproduces the noncommutative structure $\{t, x\}_{\rm DB} = \theta$.

This analysis can be generalized trivially to higher $d+1$-dimensional Galilean space–time. In the case of $d \geqslant 2$, one can see that the above space–time noncommutativity is of the form

[2] We show (see appendix A) how this is done for a free Maxwell theory where the DB between phase-space variables in radiation and axial gauges are related by appropriate gauge transformations.

$\{x^0, x^i\}_{\rm DB} = \theta^{0i}$; $(x^0 = t)$. This can be derived by writing the transformations (25) and (26) for $d \geqslant 2$ as

$$x'^0 = x^0 + \epsilon \tag{30}$$

$$x'^i = x^i + \epsilon \frac{{\rm d}x^i}{{\rm d}\tau} = x^i + \epsilon \frac{p^i}{m}, \tag{31}$$

which, when substituted back in the LHS of $\{x'^0, x'^i\} = \theta^{0i}$, fixes ϵ as

$$\epsilon = -\theta^{0i} p_i. \tag{32}$$

The desired gauge fixing condition (dropping the prime) now becomes

$$x^0 + \theta^{0i} p_i - \tau \approx 0 \tag{33}$$

which is the analogue of (29). For $d \geqslant 2$, the space–space algebra is also NC

$$\{x^i, x^j\}_{\rm DB} = -\frac{1}{m}(\theta^{0i} p^j - \theta^{0j} p^i). \tag{34}$$

The remaining non-vanishing DB(s) are

$$\{x^i, p_0\}_{\rm DB} = -\frac{p^i}{m} \qquad \{x^i, p_j\}_{\rm DB} = \delta^i{}_j. \tag{35}$$

The above structures of the Dirac brackets show a Lie-algebraic structure for the brackets involving phase-space variables (with the inclusion of identity). Following [6], one can therefore associate an appropriate 'diamond product' for this, in order to compose any pair of phase-space functions.

We have thus systematically derived the nonstandard gauge condition leading to a noncommutative algebra. Also, the change of variables mapping this noncommutative algebra with the usual (commutative) algebra is found to be a gauge transformation.

There is another interesting way of deriving the Dirac algebra if one looks at the symplectic 2-form $\omega = {\rm d}p_\mu \wedge {\rm d}x^\mu$ and then simply imposes the conditions on p_0 and x^0, for all cases discussed. We consider the simplest case here. In 1 + 1-dimension, the 2-form ω can be written as

$$\omega = {\rm d}p_t \wedge {\rm d}t + {\rm d}p_x \wedge {\rm d}x. \tag{36}$$

Now imposing the condition on p_t (14) and t (20), we get

$$\omega = -\frac{p_x}{m}\,{\rm d}p_x \wedge {\rm d}\tau + {\rm d}p_x \wedge {\rm d}x. \tag{37}$$

Note that the first term on the right-hand side of the above equation vanishes as τ is not a variable in the configuration space. Now the inverse of the components of the 2-form yields the brackets (23).

Next we carry out the above analysis in the nonstandard gauge (29). In this case, after imposing the condition on p_x from (29), the 2-form ω reads

$$\omega = {\rm d}p_t \wedge {\rm d}t - \frac{1}{\theta}\,{\rm d}t \wedge {\rm d}x. \tag{38}$$

Once again, a straightforward computation of the inverse of the components of the 2-form yields the noncommutative structure $\{t, x\} = \theta$. The same procedure can be followed for the other cases discussed in the paper.

The role of integral curves within this symplectic formalism [8] is discussed in appendix B.

3. Relativistic free particle

In this section we take up the case of a free relativistic particle and study how space–time noncommutativity can arise in this case also through a suitably modified gauge fixing condition. To that end, we start with the standard reparametrization invariant action of a relativistic free particle which propagates in $(d+1)$-dimensional 'target spacetime'

$$S_0 = -m \int \mathrm{d}\tau \sqrt{-\dot{x}^2} \tag{39}$$

with space–time coordinates x^μ, $\mu = 0, 1, \ldots, d$, the dot denoting differentiation with respect to the evolution parameter τ, and the Minkowski metric is $\eta = \mathrm{diag}(-1, 1, \ldots, 1)$. Note that here it is already in the reparametrized form with all x^μ (including $x^0 = t$) contained in the configuration space. The canonically conjugate momenta are given by

$$p_\mu = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}} \tag{40}$$

and satisfy the standard PB relations

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu \qquad \{x^\mu, x^\nu\} = \{p^\mu, p^\nu\} = 0. \tag{41}$$

These are subject to the Einstein constraint

$$\phi_1 = p^2 + m^2 \approx 0 \tag{42}$$

which follows by taking the square of (40). Now using the reparametrization symmetry of the problem (under which the action (39) is invariant) and the fact that $x^\mu(\tau)$ transforms as a scalar under world-line reparametrization (15), again leads to the infinitesimal transformation of the space–time coordinate (16). As before, to derive the generator of the reparametrization invariance we write the variation in the Lagrangian as a total derivative,

$$\delta L = \frac{\mathrm{d}B}{\mathrm{d}\tau}, \qquad B = -m\epsilon\sqrt{-\dot{x}^2}. \tag{43}$$

The generator is obtained from the usual Noether's prescription[3]

$$G = \frac{1}{2}(p^\mu \delta x_\mu - B) = \frac{\epsilon\sqrt{-\dot{x}^2}}{2m}\phi_1 \tag{44}$$

where we have used (16) and (43). Clearly we find that G generates the infinitesimal transformation of the space–time coordinate (19). Now we can impose a gauge condition to curtail the gauge freedom just as in the NR case. The standard choice is to identify the time coordinate x^0 with the parameter τ,

$$\phi_2 = x^0 - \tau \approx 0 \tag{45}$$

which is the analogue of (20). The constraints (42) and (45) form a second class set with

$$\{\phi_a, \phi_b\} = 2p_0\epsilon_{ab}. \tag{46}$$

The resulting non-vanishing DB(s) are

$$\{x^i, p_0\}_{\mathrm{DB}} = \frac{p^i}{p_0} \qquad \{x^i, p_j\}_{\mathrm{DB}} = \delta^i{}_j \tag{47}$$

which impose the constraints ϕ_1 and ϕ_2 strongly. In particular, we observe $\{x^0, x^i\}_{\mathrm{DB}} = 0$, showing that there is no space–time noncommutativity. This is again consistent with the fact

[3] The factor of $1/2$ comes from symmetrization. To make this point clear, we must note that while computing $\{x^\mu, G\}$, an additional factor of 2 crops up from the bracket between x^μ and δx_μ as δx_μ is related to p_μ by the relations (16) and (40). The factor of $1/2$ is placed in order to cancel this additional factor of 2.

that the constraint (45) is now strongly imposed. Taking a cue from our previous NR example, we see that we must have a variant of (45) as a gauge fixing condition to get space–time noncommutativity in the following form:

$$\{x'^0, x'^i\}_{\mathrm{DB}} = \theta^{0i} \tag{48}$$

(θ^{0i} being constants) where x'^μ denotes the appropriate gauge transforms of x^μ variables. To determine these transformed variables x'^μ in terms of the variables x^μ, we consider an infinitesimal transformation (16) written in terms of phase-space variables as

$$x'^0 = x^0 + \epsilon, \qquad x'^i = x^i - \epsilon \frac{p^i}{p_0} \tag{49}$$

where we have used the relation $\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = -\frac{p^i}{p_0}$ obtained from (40). Substituting the above relations (49) back in (48) and using (47), a simple inspection shows that ϵ is given by

$$\epsilon = -\theta^{0i} p_i \tag{50}$$

which is identical to (32). Hence the gauge transformed variables x'^μ (49) for the above choice of ϵ are given by

$$x'^0 = x^0 - \theta^{0i} p_i \tag{51}$$

$$x'^i = x^i + \theta^{0j} p_j \frac{p^i}{p_0}. \tag{52}$$

Using the above set of transformations and the relation (47), we obtain the Dirac algebra between the primed variables,

$$\{x'^0, x'^i\}_{\mathrm{DB}} = \theta^{0i} \tag{53}$$

$$\{x'^i, x'^j\}_{\mathrm{DB}} = \frac{1}{p_0}(\theta^{0i} p^j - \theta^{0j} p^i) \tag{54}$$

$$\{x'^i, p'_0\}_{\mathrm{DB}} = \frac{p^i}{p_0}, \qquad \{x'^i, p'_j\}_{\mathrm{DB}} = \delta^i{}_j \tag{55}$$

Note that unlike x, p are gauge invariant objects as $\{p^\mu, \phi\} = 0$; hence $p'_\mu = p_\mu$.

It is interesting to observe that the solution of the gauge parameter ϵ remains the same in both the relativistic case as well as the NR case. Also, m in the NR case gets replaced by $-p_0$ in the relativistic case. With this identification, one can easily see that the complete Dirac algebra in the NR case goes over to the corresponding algebra in the relativistic case. However, since p_0 does not have a vanishing bracket with all other phase-space variables, its occurrence in the denominators in (54) and (55) shows that the bracket structure of the phase-space variables in the relativistic case is no longer Lie-algebraic, unlike the NR case discussed in the previous section.

Furthermore, the modified gauge fixing condition is given by

$$\phi_2 = x^0 + \theta^{0i} p_i - \tau \approx 0, \qquad i = 1, 2, \ldots, d \tag{56}$$

It is trivial to check that the constraints (42) and (56) also form a second class pair as

$$\{\phi_a, \phi_b\} = 2p_0 \epsilon_{ab}. \tag{57}$$

The set of non-vanishing DB(s) consistent with the strong imposition of the constraints (42) and (56) reproduces the results (53)–(55). Equation (55) is the same as in the standard gauge (45), while (54) implies non-trivial commutation relations among spatial coordinates upon quantization.

It should be noted that the above gauge fixing condition (56) was also given in [4]. Indeed a change of variables, which is different from (51) and (52), is found there by inspection, using which the space–time noncommutativity gets removed. However, the change of variables given in this paper is related to a gauge transformation which in turn gives a systematic derivation of the modified gauge condition and also space–time noncommutativity. Moreover, their [4] definition of the Lorentz generators (rotations and boosts) requires some additional terms (in the modified gauge) in order to have a closed algebra between the generators. In our approach, the definition of the Lorentz generators remains unchanged, simply because these are gauge invariant.

The Lorentz generators (rotations and boosts) are defined as

$$M_{ij} = x_i p_j - x_j p_i \tag{58}$$

$$M_{0i} = x_0 p_i - x_i p_0. \tag{59}$$

Expectedly, they satisfy the usual algebra in both the unprimed and the primed coordinates as $M_{\mu\nu}$ and p_μ are both gauge invariant.

$$\{M_{ij}, p_k\}_{\mathrm{DB}} = \delta_{ik} p_j - \delta_{jk} p_i \tag{60}$$

$$\{M_{ij}, M_{kl}\}_{\mathrm{DB}} = \delta_{ik} M_{jl} - \delta_{jk} M_{il} + \delta_{jl} M_{ik} - \delta_{il} M_{jk} \tag{61}$$

$$\{M_{ij}, M_{0k}\}_{\mathrm{DB}} = \delta_{ik} M_{0j} - \delta_{jk} M_{0i} \tag{62}$$

$$\{M_{0i}, M_{0j}\}_{\mathrm{DB}} = M_{ji}. \tag{63}$$

However, the algebra between the space coordinates and the rotations, boosts are different in the two gauges (45) and (56). This is expected as x^k is not gauge invariant under gauge transformation. We find

$$\{M_{ij}, x^k\}_{\mathrm{DB}} = \delta_i{}^k x_j - \delta_j{}^k x_i \tag{64}$$

$$\{M_{0i}, x^j\}_{\mathrm{DB}} = x_i \frac{p^j}{p_0} - x_0 \delta_i{}^j \tag{65}$$

$$\begin{aligned} \{M_{ij}, x'^k\}_{\mathrm{DB}} &= \left\{ M_{ij}, x^k + \theta^{0l} p_l \frac{p^k}{p_0} \right\}_{\mathrm{DB}} \\ &= \delta_i{}^k x'_j - \delta_j{}^k x'_i + \frac{p^k}{p_0} \left(\theta^0{}_i p_j - \theta^0{}_j p_i \right) \end{aligned} \tag{66}$$

$$\begin{aligned} \{M_{0i}, x'^j\}_{\mathrm{DB}} &= \left\{ M_{0i}, x^j + \theta^{0l} p_l \frac{p^j}{p_0} \right\}_{\mathrm{DB}} \\ &= x'_i \frac{p^j}{p_0} - x'_0 \delta_i{}^j - \theta^0{}_i p^j \end{aligned} \tag{67}$$

where we have used the algebra (47) followed by (52). The same results can also be obtained using the relations (53)–(55).

The gauge choice (56) is not Lorentz invariant. Yet the Dirac bracket procedure forces this constraint equation to be strongly valid in all Lorentz frames [12]. This can be made consistent if and only if an infinitesimal Lorentz boost to a new frame[4]

$$p^\mu \to p'^\mu = p^\mu + \omega^{\mu\nu} p_\nu \tag{68}$$

is accompanied by a compensating infinitesimal gauge transformation

$$\tau \to \tau' = \tau + \Delta\tau. \tag{69}$$

[4] A similar treatment has been given in [10] for a free relativistic particle coupled to Chern–Simons term.

The change in x^μ, up to first order, is therefore

$$\begin{aligned} x'^{\mu}(\tau) &= x^{\mu}(\tau') + \omega^{\mu\nu}x_{\nu}(\tau) \\ &= x^{\mu}(\tau) + \Delta\tau\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} + \omega^{\mu\nu}x_{\nu}. \end{aligned} \tag{70}$$

In particular, the zeroth component is given by

$$x'^{0}(\tau) = x^{0}(\tau) + \Delta\tau\frac{\mathrm{d}x^{0}}{\mathrm{d}\tau} + \omega^{0i}x_{i}. \tag{71}$$

Since the gauge condition (56) is $x^0(\tau) \approx \tau - \theta^{0i}p_i$, $x'^0(\tau)$ also must satisfy $x'^0(\tau) = (\tau - \theta^{0i}p'_i)$ in the boosted frame, which can now be written, using (68), as

$$\begin{aligned} x'^{0}(\tau) &= \tau - \theta^{0i}p'_{i} \\ &= \tau - \theta^{0i}p_{i} + \theta^{0i}\omega^{0i}p_{0}. \end{aligned} \tag{72}$$

Comparing with (71) and using the gauge condition (56), we can now solve for $\Delta\tau$ to get

$$\Delta\tau = \frac{\theta^{0i}\omega^{0i}p_0 - \omega^{0i}x_i}{1 - \theta^{0i}\dot{p}_i}, \qquad \dot{p}_i = \frac{\mathrm{d}p_i}{\mathrm{d}\tau}. \tag{73}$$

Therefore, for a pure boost, the spatial components of (70) satisfy

$$\begin{aligned} \delta x^{j}(\tau) = x'^{j}(\tau) - x^{j}(\tau) &= \Delta\tau\frac{\mathrm{d}x^{j}}{\mathrm{d}\tau} + \omega^{j0}x_{0} \\ &= \omega^{0i}\left(x_{i}\frac{p^{j}}{p_{0}} - x_{0}\delta_{i}{}^{j} - \theta^{0i}p^{j}\right). \end{aligned} \tag{74}$$

Hence we find that (74) and (67) are consistent with each other. However, note that in the above derivation we have taken θ^{0i} to be a constant. If we take θ^{0i} to transform as a tensor, then for a Lorentz boost to a new frame, it changes as

$$\theta^{0i} \to \theta'^{0i} = \theta^{0i} + \omega^{0j}\theta^{ji} \tag{75}$$

and the entire consistency programme would fail. The (1+1)-dimensional case is special since even if we take θ^{01} to transform as a tensor, this will not affect the consistency programme as it remains invariant ($\theta'^{01} = \theta^{01}$) under Lorentz boost.

Let us now make certain observations. Although, the relations (34) and (54) are reminiscent of Snyder's algebra [7], there is a subtle difference. To see this, note that the right-hand side of these relations do not have the structure of an angular momentum operator in their differential representation (obtained by replacing p_j by $(-\mathrm{i}\partial_j)$ in contrast to Snyder's algebra.

Now in the cases where the noncommutativity takes the canonical structure ($[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu}$), the presence of non-locality is inferred from the fact that two localized functions f and g having supports within a size $\delta \ll \sqrt{\|\theta\|}$, yields a function $f \star g$ which is non-vanishing over a much larger region of size $\|\theta\|/\delta$ [1]. One therefore expects a similar qualitative feature of non-locality arising from the 'diamond product' appropriate for the Lie bracket structure of noncommutativity in the NR case also. This is further reinforced by the fact that coordinate transformations (30) and (31) involve mixing of coordinates and momenta. Since this mixing is present in the relativistic case as well (51) and (52), it is expected to maintain the non-locality of the noncommutative theory, although an appropriate 'diamond product' cannot be readily constructed because of the absence of a Lie bracket structure. Also, the mixing of coordinates and momenta is a natural consequence of our gauge conditions which essentially involve phase-space variables interpolating between the commutative and noncommutative descriptions. Note however, the transformed coordinates (30) and (31) are distinct from the covariant coordinates $\hat{X}^i = \hat{x}^i + \theta^{ij}\hat{A}_j$ (where $\hat{A}_j$ is a noncommutative gauge field) introduced in [6], at the noncommutative field theoretical level, to render the transformation property of the product $\hat{X}^i\psi$ covariant just like the field $\psi(\hat{x}^i)$. This is because $\hat{A}_i$ cannot be identified with $\hat{p}_\mu$, as at the noncommutative field theoretical level one does not have any $\hat{p}_\mu$ conjugate to $\hat{x}^\mu$ since $\hat{x}^\mu$ are just a set of operator valued q-numbers labelling the degrees of freedom in the system and are not regarded as independent configuration space variables.

Besides, space–time noncommutativity arising from a relation like (53) implies that the 'coordinate' time $\hat{x}^0$ cannot be localized as any state will have a spread in the spectrum of $\hat{x}^0$. This leads to the failure of causality and eventually violation of locality in quantum field theory [9].

4. Interaction with background electromagnetic field

In this section, we consider interactions with a background electromagnetic field which still keeps the time reparametrization symmetry of the relativistic free particle intact. Before discussing the general case, we consider a constant background field. The interaction term to be added to S_0 is then

$$S_F = -\frac{1}{2}\int \mathrm{d}\tau\ F_{\mu\nu}x^\mu\dot{x}^\nu \tag{76}$$

where $F_{\mu\nu}$ is a constant field strength tensor. The canonical momenta are given by

$$\Pi_\mu = p_\mu + \tfrac{1}{2}F_{\mu\nu}x^\nu \tag{77}$$

where p_μ is given by (40). The reparametrization symmetry again leads to the Einstein constraint (42) which is the first class constraint of the theory. The Poisson brackets are[5]

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu \qquad \{x^\mu, x^\nu\} = 0 \qquad \{p_\mu, p_\nu\} = -F_{\mu\nu}. \tag{78}$$

Note that p_μ does not have zero Poisson bracket with the constraint (42) anymore, and thus is not gauge invariant. Now to obtain the generator of reparametrization symmetry, we again exploit the infinitesimal transformation of the space–time coordinate given by (16). Proceeding exactly as in the earlier sections, we write the variation of the Lagrangian in a total derivative form as

$$\delta L = \frac{\mathrm{d}B}{\mathrm{d}\tau}, \qquad B = -m\epsilon\sqrt{-\dot{x}^2} - \frac{\epsilon}{2}F_{\mu\nu}x^\mu\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}. \tag{79}$$

Then the generator is obtained from the usual Noether's prescription (as was done for the case of the free relativistic particle), by making use of (77) to get

$$G = \frac{1}{2}(\Pi^\mu\delta x_\mu - B) = \frac{\epsilon\sqrt{-\dot{x}^2}}{2m}\phi_1 \tag{80}$$

where $\phi_1 = p^2 + m^2 \approx 0$ is the first class constraint (42). Clearly we find that G generates the infinitesimal transformation of the space–time coordinate (19). Hence we have again shown that the generator is indeed proportional to the first class constraint which is in agreement with Dirac's treatment. Also, the relation between reparametrization symmetry and gauge symmetry becomes evident. Now the gauge/reparametrization symmetry can be fixed by imposing a gauge condition. The standard choice is given by (45). The constraints (42) and (45) form a second class set with the Poisson brackets between them given by (46). So the non-vanishing Dirac brackets are given by (47) and

$$\{p_i, p_j\}_{\mathrm{DB}} = -F_{ij} \qquad \{p_0, p_i\}_{\mathrm{DB}} = F_{ij}\frac{p_j}{p_0}. \tag{81}$$

[5] These relations follow from the basic canonical algebra $\{x_\mu, \Pi^\nu\} = \delta^\nu_\mu$; $\{x_\mu, x_\nu\} = \{\Pi_\mu, \Pi_\nu\} = 0$.

To obtain noncommutativity between the primed set of space–time coordinates (48), we first observe that the zeroth component and spatial components of (16) (in the standard gauge (45)) leads to (49) where we have used the relation $\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = -\frac{p_i}{p_0}$ obtained from (40). Using the relations (48) and (49) fixes the value of ϵ, which, in view of the non-vanishing bracket (81), turns out to be

$$\epsilon = -\theta^{0j} P_j \tag{82}$$

where

$$P_\mu = p_\mu + F_{\mu\nu} x^\nu \tag{83}$$

is gauge invariant since $\{P_\mu, p_\nu\} = 0$. As a simple consistency, observe that for vanishing electromagnetic field, the solution (82) reduces to the free particle solution (50). Also note that the non-vanishing Dirac brackets involving P_μ in the standard gauge (45) are given by

$$\{x^i, P_j\}_{\mathrm{DB}} = \delta^i{}_j \qquad \{P_\mu, P_\nu\}_{\mathrm{DB}} = F_{\mu\nu} \qquad \{x^i, P_0\}_{\mathrm{DB}} = \frac{p^i}{p_0}. \tag{84}$$

Using (82) we write down the following set of transformations which relate the unprimed and primed coordinates, following from the gauge transformation (49):

$$x'^0 = x^0 - \theta^{0i} P_i \tag{85}$$

$$x'^i = x^i - \theta^{0j} P_j \frac{\mathrm{d}x^i}{\mathrm{d}\tau} = x^i + \theta^{0j} P_j \frac{p^i}{p_0} \tag{86}$$

where we have used the relation $\frac{p'_j}{p'_0} = -\frac{\mathrm{d}x'_j}{\mathrm{d}\tau}$ since $\frac{\mathrm{d}x^0}{\mathrm{d}\tau} = 1$ in the old gauge (45). From the above set of transformations and the relations (47), (81) and (84), we compute the Dirac brackets between the primed variables

$$\{x'^0, x'^i\}_{\mathrm{DB}} = \theta^{0i} \tag{87}$$

$$\begin{aligned}\{x'^i, x'^j\}_{\mathrm{DB}} &= \frac{1}{p_0}(\theta^{0i} p^j - \theta^{0j} p^i) \\ &= \frac{1}{p'_0}(\theta^{0i} p'^j - \theta^{0j} p'^i) + O(\theta^2).\end{aligned} \tag{88}$$

In order to express the variables on the RHS in terms of primed ones[6], use has been made of (86) to get

$$\frac{p'_j}{p'_0} = \frac{p_j}{p_0} - \theta^{0k} P_k \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{p_j}{p_0}\right) + O(\theta^2). \tag{89}$$

Observe that the changes of variables (85) and (86) leading to the algebra among the primed variables, are basically infinitesimal gauge transformations that are valid to first order in the reparametrization parameter ϵ. Moreover, from (82) it follows that ϵ is proportional to θ. Hence, the Dirac algebra (87) and (88) between the primed variables is also valid up to order θ. But it turns out that these results are actually exact, as is now shown.

As before, it is possible to write down the modified gauge condition from the solution (82) for ϵ as

$$\phi_2 = x^0 + \theta^{0i} P_i - \tau \approx 0, \qquad i = 1, 2, \ldots, d. \tag{90}$$

[6] Note that, since P_μ (83) is gauge invariant, $P'_\mu = P_\mu$.

The constraints (42) and (90) again form a second class set with the Poisson brackets between them being given by (46). So we recover the previous Dirac brackets (87) and (88) between space–time coordinates x^μ.

Finally we consider the relativistic free particle coupled to an arbitrary electromagnetic field. As before the action is reparametrization invariant. Here we replace (76) by

$$S_F = -\int \mathrm{d}\tau\, A_\mu(x)\dot{x}^\mu. \tag{91}$$

The choice $A_\mu = -\frac{1}{2}F_{\mu\nu}x^\nu$ for constant $F_{\mu\nu}$ reproduces the action (76). The Einstein constraint (42) and Poisson brackets (78) again follow. The canonical momenta are given by

$$\Pi_\mu = p_\mu - A_\mu \tag{92}$$

where p_μ is defined by (40). The gauge symmetry can be fixed by imposing a gauge condition. The standard choice is given by (45). The constraints (42) and (45) form a second class set with the Poisson brackets between them again given by (46). So the non-vanishing Dirac brackets are given by (47) and (81). As before, exploiting the reparametrization symmetry of the problem, the infinitesimal transformation of the space–time coordinate is given by (16) which leads to (49) in the standard gauge (45) (where we have again used the relation $\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = -\frac{p^i}{p_0}$ obtained from (40)). Demanding noncommutativity between the primed set of space–time coordinates by imposing the condition (48) and using the relations (48) and (49) leads to

$$\left\{x^0 + \epsilon, x^i - \epsilon\frac{p^i}{p_0}\right\}_{\mathrm{DB}} = \theta^{0i} \tag{93}$$

which fixes the value of ϵ to be

$$\epsilon = -\theta^{0j}p_j + O(\theta^2). \tag{94}$$

Here we are content with an expression linear in θ as a gauge invariant P_μ (counterpart of (83)) cannot be defined here.

Once again we can identify a gauge (which is the same as (56)) where we have noncommutativity between space–time coordinates. Computing the Dirac bracket between the space–time coordinates in this gauge gives

$$\{x^0, x^i\}_{\mathrm{DB}} = \frac{\theta^{0i}}{1 + \theta^{0j}F_{j\mu}\frac{p^\mu}{p_0}} \tag{95}$$

which has already been given in [4]. One can easily see that to the linear order in θ, the above result goes to (48).

5. Conclusions

We have discussed an approach whereby both space–space and space–time noncommutative structures are obtained in a particular (nonstandard gauge) in models having reparametrization invariance. These structures are obtained by calculating either Dirac brackets or symplectic brackets and the results agree. We have also shown that the noncommutative results in the nonstandard gauge and the commutative results in the standard gauge are seen to be gauge transforms of each other. In other words, equivalent physics is described by working either with the usual brackets or the noncommuting brackets. We feel our approach is conceptually cleaner and more elegant than those [4] where such changes of variables are found by inspection and apparently lack any connection with the symmetries of the problem. This leads to ambiguities in the definition of physical (gauge invariant) variables. For instance, the angular momentum operator gets modified in distinct gauges, by appropriate inclusion of extra terms, so that the closure property is satisfied. In our approach, in contrast, the angular momentum remains invariant since the change of variables is just a gauge transformation. Consequently these extra terms never appear. We feel that the present approach could be useful in illuminating the role of variable changes used for relating the commuting and noncommuting descriptions in field theory.

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Appendix A

Here we would like to demonstrate how the Dirac brackets for any pair of variables, computed for Coulomb and axial gauges, are connected through gauge transformations. For that we consider the action of free Maxwell theory

$$S = -\frac{1}{4}\int \mathrm{d}^4x\, F_{\mu\nu}F^{\mu\nu}. \tag{A.1}$$

Now the first class constraints of the theory are

$$\pi_0(x) \approx 0 \qquad \partial_i \pi_i(x) \approx 0 \tag{A.2}$$

which are responsible for generating gauge transformations. The above set of constraints can be rendered second class by gauge fixing. Let us first consider the Coulomb gauge which is given by

$$A_0 \approx 0, \qquad \partial_i A_i(x) \approx 0. \tag{A.3}$$

The Dirac bracket computed between A_i, Π_j in this gauge yields the familiar transverse delta function [11, 12];

$$\begin{aligned}\{A_i(x), \Pi_j(y)\}^{(c)}_{\mathrm{DB}} &= -\left(\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}\right)\delta(x-y) \\ &= -\delta^T_{ij}\delta(x-y)\end{aligned} \tag{A.4}$$

where the superscript c denotes the Coulomb gauge.

The corresponding DB in axial gauge $A_3 \approx 0$ and $(\Pi_3 - \partial_3 A_0) \approx 0$ [7] is given by [12]

$$\{A_i(x), \Pi_j(y)\}^{(a)}_{\mathrm{DB}} = -\delta_{ij}\delta(x-y) + \delta_{3j}\frac{\partial_i}{\partial_3}\delta(x-y). \tag{A.5}$$

Now the gauge field configurations $A_i^{(a)}$ and $A_i^{(c)}$ are connected by the gauge transformation

$$A_i^{(a)} = A_i^{(c)} + \partial_i \Lambda \tag{A.6}$$

where Λ is the gauge transformation parameter. Imposing $A_3^{(a)} = 0$ (axial gauge) fixes the value of Λ to be

$$\Lambda = -\frac{1}{\partial_3}A_3^{(c)} \tag{A.7}$$

so that

$$A_i^{(a)} = A_i^{(c)} - \frac{\partial_i}{\partial_3}A_3^{(c)}. \tag{A.8}$$

[7] This follows by demanding time conservation of the gauge; i.e., $\partial_0 A_3 = \partial_0 A_3 - \partial_3 A_0 + \partial_3 A_0 = -\Pi_3 + \partial_3 A_0 \approx 0$.

On the other hand, Π_i is gauge invariant, $\Pi_i^{(a)} = \Pi_i^{(c)}$. Hence, we have

$$\{A_i(x), \Pi_j(y)\}_{\mathrm{DB}}^{(a)} = \left\{A_i(x) - \frac{\partial_i}{\partial_3} A_3(x), \Pi_j(y)\right\}_{\mathrm{DB}}^{(c)}. \tag{A.9}$$

Using the Coulomb gauge result (A.4), the axial gauge algebra (A.5) is correctly reproduced.

Appendix B

In this appendix, we develop the symplectic formalism and show the connection between integral curves and Hamilton's equations of motion in the time-reparametrized version.

Let $Q = R \times Q_0$ ($Q_0 = q^i(t), i = 1, 2, \ldots, n$) be a $(n + 1)$-dimensional configuration space which includes time t. The corresponding phase-space Γ is $(2n + 2)$-dimensional with coordinates (t, q^i, p_t, p_i). On this phase-space, a function $F(t, q^i, p_t, p_i)$ is defined as follows:

$$F(t, q^i, p_t, p_i) = p_t + H_0(q^i, p_i). \tag{B.1}$$

Also let $\tilde{\theta} = p_t \,\mathrm{d}t + p_i \,\mathrm{d}q^i$ be a 1-form on Γ. Now let Σ be a sub-manifold of Γ defined by $F(t, q^i, p_t, p_i) = 0$. Restricting $\tilde{\theta}$ to Σ, we get

$$\tilde{\theta}|_\Sigma = -H_0(q^i, p_i)\,\mathrm{d}t + p_i \,\mathrm{d}q^i. \tag{B.2}$$

An arbitrary tangent vector $\vec{X}$ to a curve in Σ is given by

$$\vec{X} = u\frac{\partial}{\partial t} + v^j(q^i, p_i)\frac{\partial}{\partial q^j} + f_j(q^i, p_i)\frac{\partial}{\partial p_j} \tag{B.3}$$

with u, v^j and f_j being arbitrary coefficients.

Demanding that the 2-form $\tilde{\omega} = \mathrm{d}\tilde{\theta}|_\Sigma$ is degenerate, i.e., $\exists \vec{X} \neq 0$, such that upon contraction, the 1-form $\tilde{\omega}(\vec{X}) = 0$, we immediately obtain

$$f_i + u\frac{\partial H_0}{\partial q^i} = 0 \tag{B.4}$$

$$-v_i + u\frac{\partial H_0}{\partial p_i} = 0. \tag{B.5}$$

Hence (57) can be written as

$$\vec{X} = u\left(\frac{\partial}{\partial t} + \frac{\partial H_0}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H_0}{\partial q^i}\frac{\partial}{\partial p_i}\right). \tag{B.6}$$

Now recall that an integral curve of a vector field is a curve such that the tangent at any point to this curve gives the value of the vector field at that point.

In general, any tangent vector field $\vec{X}$ to a family of curves, parametrized by τ, in the space Σ can be written as

$$\vec{X} = \dot{x}^\mu \partial_\mu, \qquad \dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \dot{t}\frac{\partial}{\partial t} + \dot{q}^i\frac{\partial}{\partial q^i} + \dot{p}_i\frac{\partial}{\partial p_i}. \tag{B.7}$$

Comparing (B.6) and (B.7), the equations of the integral curves are given by

$$\dot{q}^i = u\frac{\partial H_0}{\partial p_i}, \qquad \dot{t} = u, \qquad \dot{p}_i = -u\frac{\partial H_0}{\partial q^i}. \tag{B.8}$$

Note that in the $t = \tau$ gauge, we recover the usual Hamiltonian equations of motion. It is the parameter u which is responsible for inducing the time reparametrization invariance.

Now we consider the example of a non-relativistic particle in (1+1)-dimension, the Hamiltonian of which reads

$$H_0 = \frac{p_x^2}{2m}. \tag{B.9}$$

In 1 + 1-dimension, the equations of the integral curves (B.8) can be rewritten as

$$\dot{x} = u\frac{\partial H_0}{\partial p_x} \qquad \dot{t} = u, \qquad \dot{p}_x = -u\frac{\partial H_0}{\partial x}. \tag{B.10}$$

Substituting the form of the Hamiltonian (B.9) in (B.10), we obtain

$$p_x = \frac{m\dot{x}}{\dot{t}} = m\frac{\mathrm{d}x}{\mathrm{d}t} = \text{constant} \tag{B.11}$$

which is the equation of the integral curve. Note that the above form of the canonical momentum is independent of the parameter u. This establishes a connection between the integral curve on Σ and the canonical momenta. Also from (B.1) and (B.9), we have

$$p_t + \frac{p_x^2}{2m} = 0, \tag{B.12}$$

which is nothing but the first class constraint (14) in the time-reparametrized version of the non-relativistic particle. Hence, from the integral curve, we also get the constraint of the time-reparametrized theory. The connection between the integral curves and the constraints for the other models discussed in the paper can be shown in a similar way following the above approach.

References

[1] See for example, Szabo R J 2003 *Phys. Rep.* **378** 207 (*Preprint* hep-th/0109162)
Douglas M R and Nekrasov N A 2001 *Rev. Mod. Phys.* **73** 977–1029 (*Preprint* hep-th/0106048)

[2] Nair V P and Polychronakos A P 2001 *Phys. Lett.* B **505** 267 (*Preprint* hep-th/0011172)

[3] Duval C and Horvathy P A 2000 *Phys. Lett.* B **479** 284 (*Preprint* hep-th/0002233)
Horvathy P A 2002 *Ann. Phys., NY* **299** 128 (*Preprint* hep-th/0201007)

[4] Pinzul A and Stern A 2004 *Phys. Lett.* B **593** 279 (*Preprint* hep-th/0402220)

[5] Banerjee R 2002 *Mod. Phys. Lett.* A **17** 631 (*Preprint* hep-th/0106280)

[6] Madore J, Schraml S, Schupp P and Wess J 2000 *Eur. Phys. J.* C **16** 161 (*Preprint* hep-th/0001203)

[7] Snyder H S 1947 *Phys. Rev.* **71** 38
Snyder H S 1947 *Phys. Rev.* **72** 68

[8] Sudarshan E C G and Mukunda N 1974 *Classical Dynamics: A Modern Perspective* (New York: Wiley)

[9] Chaichian M, Demichev A, Presnajder P and Tureanu A 2001 *Eur. Phys. J.* C **20** 767 (*Preprint* hep-th/0007156)
Balachandran A P, Govindarajan T R, Molina C and Teotonio-Sobrinho P 2004 *J. High Energy Phys.* JHEP10(2004)072 (*Preprint* hep-th/0406125)

[10] Chakraborty B 1995 *Ann. Phys., NY* **244** 312–39

[11] Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Belfer Graduate School of Science, Yeshiva University)
Sudarshan E C G and Mukunda N 1974 *Classical Dynamics: A Modern Perspective* (New York: Wiley)

[12] Hanson A, Regge T and Teitelboim C 1974 *Constrained Hamiltonian Systems* (Rome: Accademia Nazionale Dei Lincei)